Pergamon

Computers Math. Applic. Vol. 28, No. 1–3, pp. 37–43, 1994

0898-1221/94 $7.00 + 0.00

0898-1221(94)00091-3

The Rational Recursive Sequence $x_{n+1} = (\beta x_n^2)/(1 + x_{n-1}^2)$

E. CAMOUZIS AND G. LADAS
Department of Mathematics
University of Rhode Island, Kingston, RI 02881-0816, U.S.A.

I. W. RODRIGUES
Department of Mathematics
Cleveland State University, Cleveland, OH 44115, U.S.A.

S. NORTHSHIELD
Department of Mathematics
State University of New York, College at Plattsburgh, Plattsburgh, NY 72901, U.S.A.

Abstract—We investigate the behavior of solutions of the equation in the title under the hypotheses that β is a positive constant and the initial conditions x_{-1} and x_0 are arbitrary positive numbers.

Keywords—Recursive sequence, Local asymptotic stability, Global asymptotic stability.

1. INTRODUCTION

Consider the rational recursive sequence

$$x_{n+1} = \frac{\beta x_n^2}{1 + x_{n-1}^2}, \qquad n = 0, 1, \dots \tag{1.1}$$

where

$$\beta \in (0, \infty) \tag{1.2}$$

and the initial conditions x_{-1} and x_0 are arbitrary positive numbers. Our aim in this paper is to investigate the global behavior of solutions of equation (1.1). For some related results, see [1–6].

DEFINITION 1.1. STABILITY. *The equilibrium point $\bar{x}$ of equation (1.1) is called* **locally stable** *if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x_{-1} - \bar{x}| < \delta$ and $|x_0 - \bar{x}| < \delta$ implies that $|x_n - \bar{x}| < \epsilon$ for all $n \geq 0$. Otherwise, $\bar{x}$ is said to be* **unstable**.

The equilibrium point $\bar{x}$ of equation (1.1) is called **locally asymptotically stable** *if it is locally stable and there exists $\gamma > 0$ such that $|x_{-1} - \bar{x}| < \gamma$ and $|x_0 - \bar{x}| < \gamma$ imply that $\lim_{n\to\infty} |x_n - \bar{x}| = 0$.*

The equilibrium point $\bar{x}$ of equation (1.1) is called **a global attractor** *if $\lim_{n\to\infty} x_n = \bar{x}$ for all $x_{-1}, x_0 \in (0, \infty)$.*

The equilibrium point $\bar{x}$ of equation (1.1) is called **globally asymptotically stable** *if $\bar{x}$ is both locally asymptotically stable and a global attractor.*

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. LOCAL STABILITY

Equation (1.1) may have one, two, or three equilibrium points. If $0 < \beta < 2$, equation (1.1) has only one equilibrium, namely,

$$\bar{x}_1 = 0.$$

The linearized equation about the zero equilibrium is

$$y_{n+1} = 0,$$

and so is locally asymptotically stable. In Section 3, we will prove that $\bar{x}_1 = 0$ is also a global attractor of all positive solutions. Therefore, when $0 < \beta < 2$, the zero equilibrium of equation (1.1) is globally asymptotically stable.

When $\beta > 2$, equation (1.1) has three equilibria, namely,

$$\bar{x}_1 = 0, \quad \bar{x}_2 = \frac{\beta - \sqrt{\beta^2 - 4}}{2}, \quad \text{and} \quad \bar{x}_3 = \frac{\beta + \sqrt{\beta^2 - 4}}{2}.$$

As before, $\bar{x}_1 = 0$ is locally asymptotically stable.

The linearized equations about $\bar{x}_2$ and $\bar{x}_3$ are, respectively,

$$w_{n+1} - 2w_n + \frac{2\bar{x}_i}{\beta} w_{n-1} = 0 \qquad \text{for} \quad i = 2, 3$$

with respective characteristic equations

$$\lambda^2 - 2\lambda + \frac{2\bar{x}_i}{\beta} = 0 \qquad \text{for} \quad i = 2, 3.$$

From this, it follows that both $\bar{x}_2$ and $\bar{x}_3$ are unstable. More precicely, $\bar{x}_2$ is a saddle point and $\bar{x}_3$ is a repellor.

When $\beta = 2$, equation (1.1) has two equilibria, namely,

$$\bar{x}_1 = 0 \quad \text{and} \quad \bar{x}_2 = 1.$$

As in the previous cases, $\bar{x}_1 = 0$ is locally asymptotically stable.

The linearized equation of equation (1.1) about the positive equilibrium $\bar{x} = 1$ is

$$y_{n+2} - 2y_{n+1} + y_n = 0.$$

Clearly linearized stability analysis fails in this case. However, we can show directly from equation (1.1) that the equilibrium $\bar{x} = 1$ is unstable. To this end, it suffices to show that any solution of equation (1.1) with $\beta = 2$ and with

$$0 < x_0 < x_{-1} \leq 1$$

decreases monotonically to zero. Indeed

$$x_1 = \frac{2x_0^2}{1 + x_{-1}^2} \leq \frac{2x_0^2}{2x_{-1}} = x_0 \frac{x_0}{x_{-1}} < x_0.$$

(Note that $x_0 \leq x_{-1} < 1$ also implies $x_1 < x_0$.) Inductively we see that

$$x_{n+1} < x_n \qquad \text{for } n \geq 1,$$

which implies that $\{x_n\}$ decreases monotonically to zero.

3. GLOBAL ATTRACTIVITY

First we will show that every solution of equation (1.1) is bounded from above. In fact, the following result is true.

LEMMA 3.1. *Let $\{x_n\}$ be a solution of equation (1.1). Then*

$$x_n < \beta^5 \qquad \text{for } n \geq 3.$$

PROOF. Observe that for $n \geq 0$,

$$x_{n+3} = \frac{\beta x_{n+2}^2}{1 + x_{n+1}^2} = \frac{\beta^7 x_n^8}{(1+x_n^2)^2(1+x_{n-1}^2)^2[(1+x_{n-1}^2)^2 + \beta^2 x_n^4]} < \frac{\beta^7 x_n^8}{x_n^4 \beta^2 x_n^4} = \beta^5.$$

CASE $\beta > 2$. When $\beta > 2$, equation (1.1) has 3 equilibria, namely

$$\bar{x}_1 = 0, \qquad \bar{x}_2 = \frac{\beta - \sqrt{\beta^2 - 4}}{2}, \qquad \bar{x}_3 = \frac{\beta + \sqrt{\beta^2 - 4}}{2}.$$

In Section 2, we saw that $\bar{x}_1$ is locally asymptotically stable and that $\bar{x}_2$ and $\bar{x}_3$ are unstable.

Before we state the next theorem, we summarize in the following lemma a few observations about the nontrivial solutions of equation (1.1), when $\beta > 2$.

LEMMA 3.2. *Assume that $\beta > 2$ and suppose that $\{x_n\}$ is a nontrivial solution of equation (1.1). Then the following statements are true.*

(a) *If for some nonnegative integer n_0,*

$$x_{n_0+1} \leq x_{n_0} \leq \bar{x}_2 \qquad \text{or} \qquad x_{n_0+1} \leq \bar{x}_2 \leq x_{n_0},$$

then

$$x_{n+2} < x_{n+1} < \bar{x}_2 \qquad \text{for } n \geq n_0.$$

(b) *If for some nonnegative integer n_0,*

$$\bar{x}_2 \leq x_{n_0} \leq x_{n_0+1} \leq \bar{x}_3,$$

then

$$x_{n_0+2} > x_{n_0+1}.$$

(c) *If for some nonnegative integer n_0,*

$$\bar{x}_3 \leq x_{n_0+1} \leq x_{n_0},$$

then

$$x_{n_0+2} < x_{n_0+1}.$$

PROOF.

(a) Clearly

$$\beta x_{n_0+1} \leq 1 + x_{n_0+1}^2,$$

and so

$$x_{n_0+2} = \frac{\beta x_{n_0+1}^2}{1 + x_{n_0}^2} \leq \frac{\beta x_{n_0+1}^2}{1 + x_{n_0+1}^2} < \frac{\beta x_{n_0+1}^2}{\beta x_{n_0+1}} = x_{n_0+1},$$

and the result follows by induction.

(b) We have

$$\beta x_{n_0+1} \geq 1 + x_{n_0+1}^2,$$

and so

$$x_{n_0+2} = \frac{\beta x_{n_0+1}^2}{1 + x_{n_0}^2} \geq \frac{\beta x_{n_0+1}^2}{1 + x_{n_0+1}^2} > x_{n_0+1}.$$

(c) Clearly

$$\beta x_{n_0+1} \leq 1 + x_{n_0+1}^2,$$

and so

$$x_{n_0+2} = \frac{\beta x_{n_0+1}^2}{1 + x_{n_0}^2} \leq \frac{\beta x_{n_0+1}^2}{1 + x_{n_0+1}^2} < x_{n_0+1}.$$

The proof is complete.

THEOREM 3.1. *Assume that $\beta > 2$ and let $\bar{x}_2$ denote the smallest positive equilibrium of equation (1.1). Then equation (1.1) has a solution $\{x_n\}$ which is strictly increasing to $\bar{x}_2$.*

PROOF. For $x \in [0, \infty)$, set

$$f_{-1}(x) = x^2, \quad f_0(x) = x,$$

and

$$f_{n+1} = \frac{\beta f_n^2(x)}{1 + f_{n-1}^2(x)}, \qquad n = 0, 1, \dots \tag{3.1}$$

For each $x \in [0, \infty)$, the sequence of positive numbers $\{f_n(x)\}$ is a solution of equation (1.1), and by Lemma 3.1 this sequence is bounded. Let

$$s(x) = \sup_n f_n(x)$$

and

$$A = \{x \in [0, \infty) : s(x) < \bar{x}_2\}.$$

We now claim that the function s is continuous on A and that set A is open. Indeed for every $x \in A$, there exists an integer N such that

$$s(x) = f_N(x) \quad \text{and} \quad f_{N+1}(x) < f_N(x).$$

Choose

$$0 < \epsilon < \min\left\{\bar{x}_2 - f_N(x), \frac{f_N(x) - f_{N+1}(x)}{2}\right\}.$$

From the continuity of $f_{-1}, f_0, \dots, f_{N+1}$, there exists $\delta > 0$ such that if

$$|x - x'| < \delta, \qquad \text{then} \sup_{n \leq N+1} |f_n(x) - f_n(x')| < \epsilon.$$

Note that

$$f_{N+1}(x') < f_{N+1}(x) + \epsilon < f_N(x) - \epsilon < f_N(x') < f_N(x) + \epsilon < \bar{x}_2,$$

which implies that

$$s(x') = \sup_{n \leq N} f_n(x').$$

Then

$$s(x) - \epsilon = f_N(x) - \epsilon < f_N(x') \leq \sup_{n \leq N} f_n(x) = s(x'),$$

and so

$$s(x') < \sup_{n \le N} f_n(x) + \epsilon = s(x) + \epsilon < s(x) + \bar{x}_2 - f_N(x) = \bar{x}_2,$$

from which it follows, as we claimed, that s is continuous on A and that the set A is open.

Set $\lambda = \sup A$. Then, since A is open, λ is not in A and

$$s(\lambda) \ge \bar{x}_2.$$

But $s(\lambda) \le \bar{x}_2$, and so

$$s(\lambda) = \bar{x}_2.$$

Hence, by Lemma 3.2 (a),

$$\lim_{n\to\infty} f_n(\lambda) = \bar{x}_2.$$

The proof is complete.

THEOREM 3.2. *Assume that $\beta > 2$ and let $\bar{x}_2$ denote the smallest positive equilibrium of equation (1.1). Then there exists a solution $\{x_n\}$ of equation (1.1) such that*

$$x_{n-1} > \bar{x}_2, \qquad n = 0, 1, \ldots$$

PROOF. Assume for the sake of contradiction, that equation (1.1) has no solution which is above $\bar{x}_2$.

For $x \in (\bar{x}_2, \infty)$, define the sequence of continuous functions

$$f_{-1}(x) = x, \quad f_0(x) = x^2$$

and

$$f_{n+1}(x) = \frac{\beta f_n^2(x)}{1 + f_{n-1}^2(x)}, \qquad n = 0, 1, \ldots$$

For each $x \in (\bar{x}_2, \infty)$, the sequence of positive numbers $\{f_n(x)\}$ is a solution of equation (1.1), and by Lemma 3.1 this sequence is bounded.

Let

$$s(x) = \sup_n f_n(x).$$

In view of the hypothesis that equation (1.1) has no solution which is above $\bar{x}_2$, it follows that for every $x \in (\bar{x}_2, \infty)$, there exist two integers m and k (which depend on x), with $m \ge -1$ and $k \ge m$, such that

$$s(x) = \sup_{-1 \le n \le m} f_n(x),$$

$$f_n(x) > f_{n+1}(x) \qquad \text{for } n \ge m,$$

and

$$f_k(x) < \bar{x}_2.$$

Choose

$$0 < \epsilon < \min_{m \le n \le k} \left\{ \frac{f_n(x) - f_{n+1}(x)}{2}, \bar{x}_2 - f_k(x) \right\}.$$

By the continuity of $f_{-1}, f_0, \ldots, f_k$, there exists $\delta > 0$ such that

$$|x' - x| < \delta \quad \text{implies} \quad \sup_{-1 \le n \le k} |f_n(x) - f_n(x')| < \epsilon.$$

Note that

$$f_n(x') > f_n(x) - \epsilon > f_{n+1}(x) + \epsilon > f_{n+1}(x') \qquad \text{for } n = m, \ldots, k$$

and

$$f_k(x') < f_k(x) + \epsilon < \bar{x}_2,$$

and so

$$s(x') = \sup_{-1 \le n \le m} f_n(x').$$

Therefore,

$$s(x) - \epsilon = \sup_{-1 \le n \le m} f_n(x) - \epsilon$$

$$< \sup_{-1 \le n \le m} f_n(x') = s(x') < \sup_{-1 \le n \le m} f_n(x) + \epsilon = s(x) + \epsilon,$$

and so

$$|s(x') - s(x)| < \epsilon,$$

which shows that the function s is continuous on $(\bar{x}_2, \infty)$.

As

$$s(1) > \bar{x}_3 \qquad \text{and} \qquad s(\sqrt{\bar{x}_2}\,) = \sqrt{\bar{x}_2} < \bar{x}_3,$$

it follows that there exists an $x^* \in (\sqrt{\bar{x}_2}, 1)$ such that

$$s(x^*) = \bar{x}_3.$$

Hence, there exists an m such that

$$\bar{x}_3 = s(x^*) = \sup_{-1 \le n \le m} f_n(x^*).$$

Also note that

$$(s(x^*) - f_{-1}(x^*))(s(x^*) - f_0(x^*)) \neq 0,$$

and so we can assume that

$$\bar{x}_3 = s(x^*) = f_N(x^*) \qquad \text{for } 1 \le N \le m,$$

$$f_{N+1}(x^*) < \bar{x}_3, \qquad \text{and} \qquad f_{N-1}(x^*) < \bar{x}_3.$$

But then

$$\bar{x}_3 > f_{N+1}(x^*) = \frac{\beta \bar{x}_3^2}{1 + f_{N-1}^2(x)} > \bar{x}_3,$$

which is a contradiction. The proof is complete.

CASE $\beta = 2$. When $\beta = 2$, equation (1.1) becomes

$$x_{n+1} = \frac{2x_n^2}{1 + x_{n-1}^2}, \qquad n = 0, 1, \dots \tag{3.2}$$

Clearly, equation (3.2) has the identically equal to 1 solution, and unless $x_{-1} = x_0 = 1$, no solution can eventually become identically equal to 1. Also by Lemma 3.1, every solution of equation (3.2) is bounded from above.

The following two theorems describe the asymptotic behavior of all solutions of equation (3.2).

THEOREM 3.3. *Equation* (3.2) *has a solution which is strictly increasing to 1.*

PROOF. The proof is similar to that of Theorem 3.1 and will be omitted.

THEOREM 3.4. *Assume that $\{x_n\}$ is a solution of equation (3.2) which is neither identically equal to one nor strictly increasing to 1. Then*

$$\lim_{n \to \infty} x_n = 0.$$

PROOF. Note that 1 is the only positive equilibrium of equation (1.1). Also, $\{x_n\}$ is a positive and bounded sequence. Hence, there exists an $n_0 \geq 0$ such that

$$x_{n_0} \leq x_{n_0-1}.$$

(Otherwise $\{x_n\}$ would be strictly increasing to 1 or to ∞.) Now observe that

$$x_{n_0+1} = \frac{2x_{n_0}^2}{1+x_{n_0-1}^2} \leq x_{n_0},$$

and by induction,

$$x_{n+1} \leq x_n \qquad \text{for } n \geq n_0.$$

Hence,

$$L = \lim_{n\to\infty} x_n$$

exists and either $L = 0$ or $L = 1$. Assume for the sake of contradiction that $L = 1$. Set

$$y_n = \frac{x_{n+1}}{x_n} \qquad \text{for } n \geq n_0.$$

Then for $n \geq n_0$, $0 < y_n \leq 1$ and

$$y_{n+1} = \frac{x_{n+2}}{x_{n+1}} = \frac{2x_{n+1}}{1+x_n^2} \leq \frac{2x_{n+1}}{2x_n} = y_n.$$

Also

$$\lim_{n\to\infty} y_n = \frac{L}{L} = 1.$$

Hence, $y_n = 1$ for $n \geq n_0$, which implies that $x_{n+1} = x_n$ for $n \geq n_0$. Therefore, $x_n = 1$ for $n \geq n_0$ which is true only if $x_{-1} = x_0 = 1$. This is a contradiction and the proof is complete.

CASE $\beta < 2$. When $0 < \beta < 2$, then equation (1.1) has only the zero equilibrium. In Section 2, we saw that this equilibrium is locally asymptotically stable. The following theorem shows that in this case the zero equilibrium is a global attractor of all positive solutions of equation (1.1).

THEOREM 3.5. *Assume that $0 < \beta < 2$ and let $\{x_n\}$ be a solution of equation (1.1). Then*

$$\lim_{n\to\infty} x_n = 0.$$

PROOF. As we proved in Lemma 3.1, every solution of equation (1.1) is bounded from above by a positive constant. Therefore there exists an $n_0 \geq 0$ such that

$$x_{n_0-1} \geq x_{n_0}.$$

Then,

$$x_{n_0+1} = \frac{\beta x_{n_0}^2}{1+x_{n_0-1}^2} \leq \frac{\beta x_{n_0}^2}{2x_{n_0-1}} \leq \frac{\beta x_{n_0}^2}{2x_{n_0}} = \frac{\beta}{2} x_{n_0},$$

and so by induction,

$$x_{n_0+n} \leq \left(\frac{\beta}{2}\right)^n x_{n_0} \qquad \text{for } n \geq 1,$$

from which the result follows.

REFERENCES

1. R. De Vault, V.L. Kocic and G. Ladas, Global stability of a recursive sequence, *Dynamic Systems and Appl.* **1**, 13–21 (1992).
2. J. Jaroma, G. Ladas and S.A. Kuruklis, Oscillations and stability in a discrete delay logistic model, *Ukrain. Math. J.* **43**, 734–744 (1991).
3. V.L. Kocic, G. Ladas and I.W. Rodrigues, On rational recursive sequences, *J. Math. Anal. Appl.* **173**, 127–157 (1993).
4. V.L. Kocic and G. Ladas, Global attractivity in nonlinear delay difference equations, *Proc. Amer. Math. Soc.* **115**, 1083–1088 (1992).
5. V.L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, (1993).
6. S.A. Kuruklis and G. Ladas, Oscillation and global attractivity in a discrete delay logistic model, *Quart. Appl. Math. L.*, 227–233 (1992).